# Section 12.3:  The Dot Product

# Things we'll go over today…

Section 12.3: The Dot Product

- Definition of the dot product
- Properties of the dot product
- Geometric interpretation of the dot product
- When are 2 vectors orthogonal?
- Direction angles and direction cosines
- Scalar and vector projections
- Work (when $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{d}}$ are constant, but not necessarily in the same direction)

# 1. Definition of the Dot Product

Def: Given (2-dimensional) vectors $\vec{v} = < a, b >$ and $\vec{w} = < c, d >$, their dot product (notation $\vec{v} \cdot \vec{w}$) is the scalar given by …

$$< a, b > \cdot < c, d > \equiv \; ac + bd$$

Ex 1: Find the following …

a) $< 2, -4 > \cdot < 3, 2 >$

b) $(-5\boldsymbol{i} + 3\boldsymbol{j}) \cdot (9\boldsymbol{i} - 7\boldsymbol{j})$

# 1. Definition of the Dot Product

Def: Given (3-dimensional) vectors $\vec{v} = < a, b, c >$ and $\vec{w} = < d, e, f >$, their dot product (notation $\vec{v} \cdot \vec{w}$) is the scalar given by …

$$< a, b, c > \cdot < d, e, f > \ \equiv \ ad + be + cf$$

Ex 2: Find the following …

a) $< -1, 4, 7 > \cdot < 2, -3, -2 >$

b) $(-2\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}) \cdot (4\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

# 2. Properties of the Dot Product

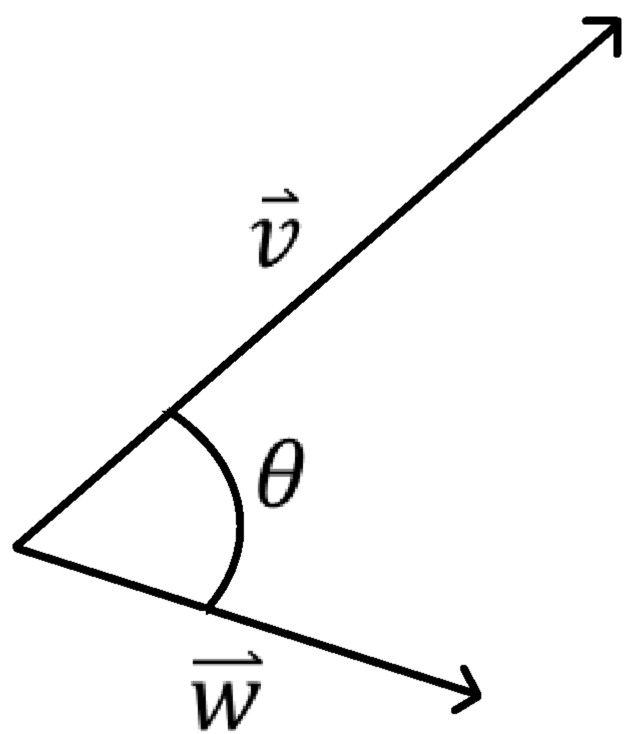
**2 Properties of the Dot Product** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $V_3$ and $c$ is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

<u>Prove some of these </u>…

# 3. Geometric Interpretation of the Dot Product

Def: The angle between vectors $\vec{v}$ and $\vec{w}$ is the smaller of the 2 angles between the vectors when the vectors are drawn tail-to-tail.



# 3. Geometric Interpretation of the Dot Product

**3 Theorem** If $\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

<u>Proof</u>: …

# 3. Geometric Interpretation of the Dot Product

Ex 3: If vectors $\vec{v}$ and $\overrightarrow{w}$ have length 8 and 3 respectively, and the angle between these vectors is $\frac{\pi}{4}$, find $\vec{v} \cdot \overrightarrow{w}$

Ex 4: Find the angle between the vectors $\vec{v} =< 4, 3, -1 >$ and $\overrightarrow{w} =< 2, -5, 2 >$

# 4. When are 2 Vectors Orthogonal?

Def: Vectors $\vec{v}$ and $\vec{w}$ are orthogonal is the angle between them is $\frac{\pi}{2}$

Note: The zero vector $\mathbf{0}$ is considered perpendicular to all other vectors.

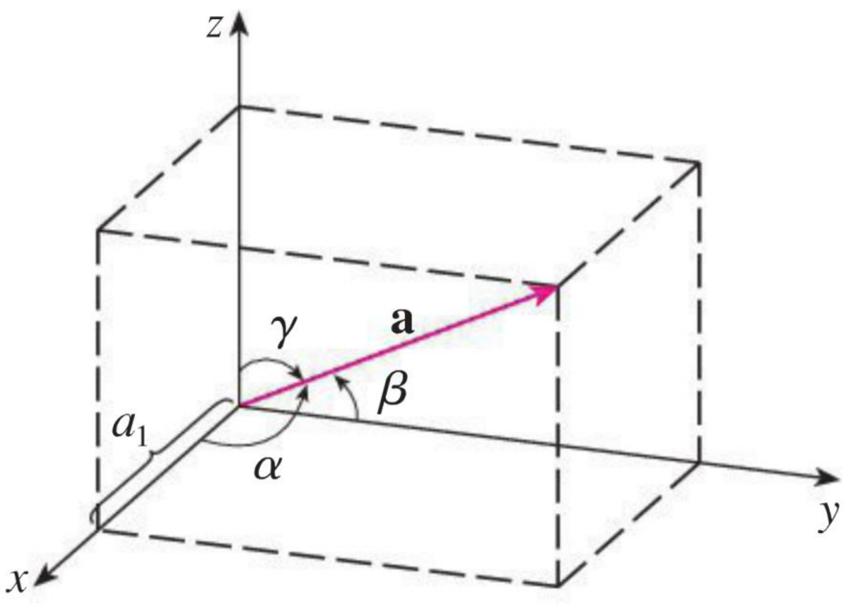
**7** Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Ex 5: a) Are the vectors $\vec{v} =< 2, 5, -2 >$ and $\vec{w} =< 3, 2, 8 >$ orthogonal?

b) Are the vectors $\vec{a} = -3\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$ and $\vec{b} = 7\boldsymbol{i} - 2\boldsymbol{k}$ orthogonal?

# 5. Direction Angles & Direction Cosines

Def: Suppose a nonzero vector $\vec{a}$ is drawn with its tail at the origin. Then the (smaller of the) angles the vector makes with the positive *x*-axis, positive *y*-axis, and positive *z*-axis are called the direction angles of the vector $\vec{a}$. These angles are denoted by $\alpha$, $\beta$, and $\gamma$ respectively.




$\cos(\alpha)$ , $\cos(\beta)$ , and $\cos(\gamma)$ are called the direction cosines of the vector $\vec{a}$

# 5. Direction Angles & Direction Cosines

<u>Some formulas involving the direction angles and direction cosines of a vector</u>: Let $\vec{a} =< a_1, a_2, a_3 >$ be a vector with direction angles $\alpha$, $\beta$, and $\gamma$. Then…

$$\cos(\alpha) = \frac{a_1}{|a|} \qquad \cos(\beta) = \frac{a_2}{|a|} \qquad \cos(\gamma) = \frac{a_3}{|a|}$$

$$cos^2(\alpha) + cos^2(\beta) + cos^2(\gamma) = 1$$

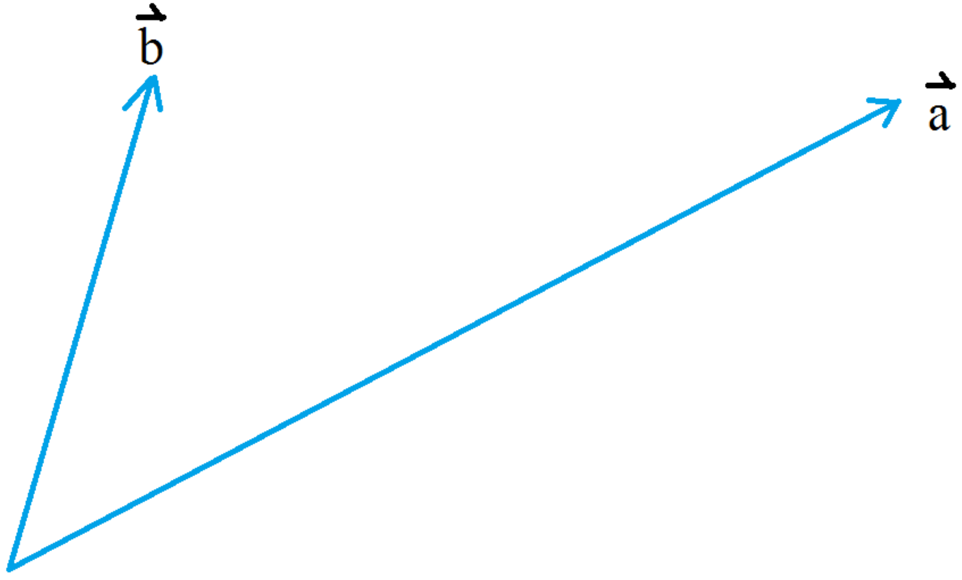
$$\frac{1}{|a|}\vec{a} =< \cos(\alpha), \cos(\beta), \cos(\gamma) >$$

# 5. Direction Angles & Direction Cosines

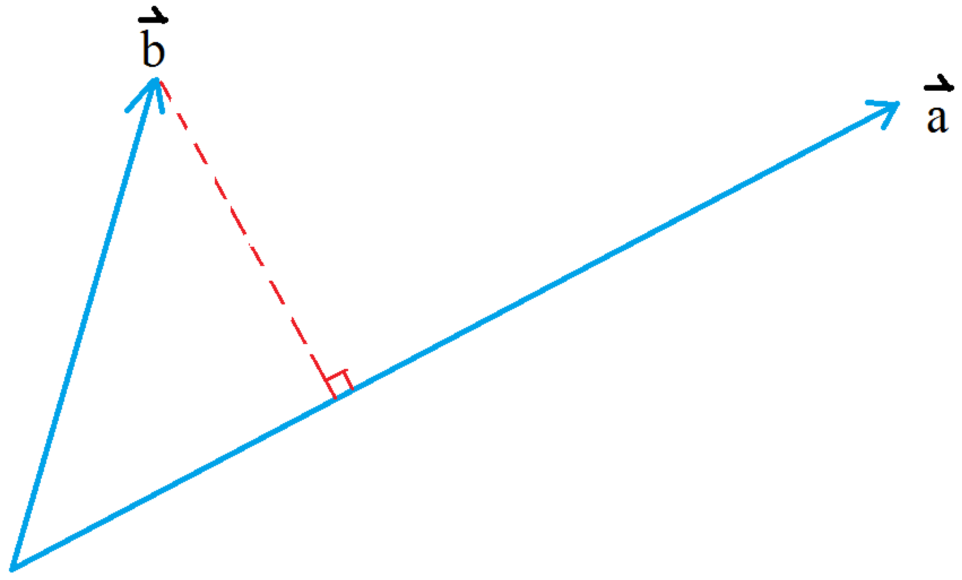
Ex 6: Find the direction angles for the vector

$$\vec{v} = < 2, -5, 1 >$$

# 6. Scalar and Vector Projections



# 6. Scalar and Vector Projections



We will do WORK when we go over section 6.4 (next class)